

# Answer to question # 55

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**Answer to question # 55 [ "Are there pictorial examples that distinguish covariant and contravariant vectors ?" D. Neuenschwander, Am. J. Phys. 65 (1), 11 (1997)]**

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Neuenschwander <sup>1</sup> asked how to visualize the distinction between co- and contravariant vectors. Most of all textbooks introduce this distinction on an abstract level, the only exception I know is Stephani <sup>2</sup>, and below I will show how I present it in my lectures "Introduction to differential geometry" at Potsdam university.

If *no metric exists* at all, then covariant vectors and contravariant vectors are different types of objects.

If *a metric exists*, then there is a canonical isomorphism between them; so we introduce *vectors*, and after fixing a coordinate system, we speak about their covariant and their contravariant components.

In the following, we will deal with the second case only, because it is more easy to visualize: The chalkboard has a canonical metric which makes it a flat two-dimensional Riemannian manifold.

Neuenschwander <sup>1</sup> wrote that the mentioned distinction is necessary when dealing with curved spaces. This is not wrong, but it is a little bit misleading, and I prefer to say: "... is necessary when dealing with a non-rectangular coordinate system." Example: We fix a point (the "origin"  $O$ ) in the Euclidean plane, then there is a one-to-one correspondence between points and

vectors. (The point  $P$  is related to the vector  $\overline{OP}$ .) First we use rectangular coordinates. We might call them  $x$  and  $y$ ; however, as we are interested to see, how the situation changes by introducing non-rectangular ones, we call them  $x^i$  with  $i \in \{1, 2\}$ . So the point  $P$  has coordinates  $(x^1, x^2)$ , cf. Fig. 1.

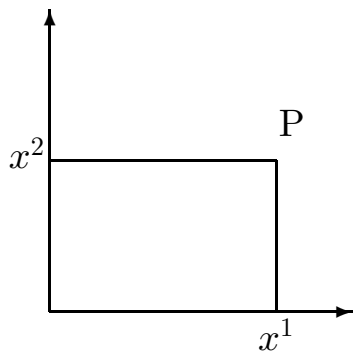


Figure 1

The coordinate system is a rectangular one, and so the component  $x^1$  can be equivalently described as the perpendicular projection to the  $x^1$ -axis or as projection parallel to the  $x^2$ -axis.

Let us now consider the case of an inclined system. Let the angle between the axes be  $\alpha$  with  $0 < \alpha < \pi$ .

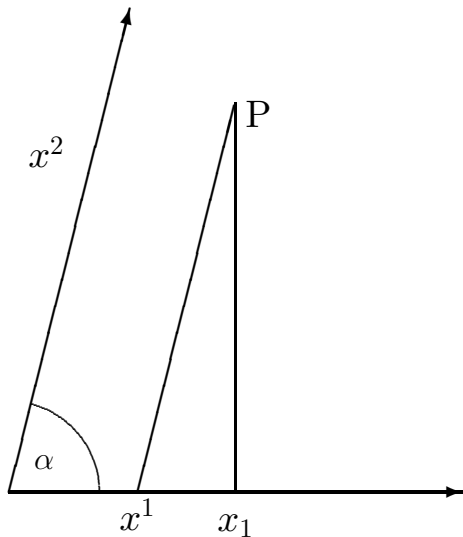


Figure 2

$x^1$  is the projection parallel to the  $x^2$ -axis, and  $x_1$  is the perpendicular projection to the  $x^1$ -axis. We get  $x_1 = x^1 + x^2 \cos \alpha$ , i.e.,  $x_1 = x^1$  if and only if  $\alpha = \pi/2$ . In general we get the following linear relation

$$x_i = g_{ij} x^j$$

by the use of the metric  $g_{ij}$ , where  $g_{12} = g_{21} = \cos \alpha$ ,  $g_{11} = g_{22} = 1$ , and summation over  $j \in \{1, 2\}$  is automatically assumed.

## References

- [1] D. Neuenschwander, Am. J. Phys. 65 (1), 11 (1997).
- [2] H. Stephani, General Relativity, Cambridge University Press 2nd edition 1990, page 26. (In the first German edition, which appeared in Berlin in 1977, it is page 35.)